

Multivariate stratified sampling by stochastic multiobjective optimisation

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Abstract

This work considers the allocation problem for multivariate stratified random sampling as a problem of integer non-linear stochastic multiobjective mathematical programming. With this goal in mind the asymptotic distribution of the vector of sample variances is studied. Two alternative approaches are suggested for solving the allocation problem for multivariate stratified random sampling. An example is presented by applying the different proposed techniques.

Key Words: Multivariate stratified random sampling, multiobjective E-model, stochastic multiobjective programming, optimum allocation, integer programming, goal programming, multiobjective V-model, multiobjective P-model.

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1 Introduction

The theory of probabilistic sampling is one of the topics of statistical theory that is most commonly used in many fields of scientific investigation. An effective survey technique in a population can be seen as an appropriate extraction of useful data which provides meaningful knowledge of the important aspects of the population. Stratified random sampling is one of the classical methods of the theory of probabilistic sampling for obtaining such information. The computation of the stratum sample size in stratified random sampling can be computed by diverse approaches, but optimum allocation, according to some criteria, has been found to be a useful approach. Optimum allocation has been stated as a non-linear mathematical programming problem in which the objective function is the variance subject to a cost restriction, or vice versa. Typically, this problem has been solved by using the Cauchy-Schwarz (Stuart, 1954) inequality, cited in Cochran (1977) or Lagrange's multiplier method, see Sukhatme *et al.* (1984).

Classical survey theory considers a single decision variable or characteristic; for example, in our case, univariate stratified random sampling studies one characteristic, the sample size and its strata allocation, see Cochran (1977), Sukhatme *et al.* (1984) and Thompson

(1997). Moreover, in the context of stratified random sampling, diverse multivariate approaches have been proposed whereby the sample size and its allocation within strata take into account several characteristics, see Sukhatme *et al.* (1984) and Arthanari and Dodge (1981). A detailed study of this problem is given by Díaz-García and Ulloa (2008).

In univariate and multivariate stratified random sampling, when the optimum allocation is performed, and the cost function is the objective function, subject to certain variance restrictions in the different characteristics, then the problem can be reduced to a classical mathematical programming problem, and for this purpose there are two well-known approaches: see Cochran (1977), Sukhatme *et al.* (1984) for the univariate case and see Arthanari and Dodge (1981) for the multivariate one, both from a deterministic point of view; and see Díaz-García and Garay (2007) and Prékopa (1978), for a stochastic approach, respectively. Observe that, in the latter case (stochastic multivariate case with the cost function as the objective function) the problem can be solved by using any of the techniques presented in Díaz-García and Garay (2007), among many others.

Alternatively, if the interest is to minimise the variances subject to a cost function, or to a given sample size, then several approaches can be adopted to solve this, for the univariate case see Cochran (1977), Thompson (1997) and Sukhatme *et al.* (1984); and see Sukhatme *et al.* (1984) and Díaz-García and Ulloa (2008) for the multivariate case. Díaz-García and Ulloa (2008) show that all the previously published approaches in this area are particular cases of the multiobjective mathematical programming technique, they proposed a unified theory for solving the problem of optimum allocation in multivariate stratified random sampling. Furthermore, Díaz-García and Ulloa (2008) propose the optimum allocation in multivariate stratified random sampling as a nonlinear problem of integer matrix mathematical programming constrained by a cost function or by a given sample size. Also, by defining a particular vectorial function of the objective function of the matrix mathematical programming problem, they show that the optimum allocation in multivariate stratified random sampling also can be studied as a non-linear multiobjective integer mathematical programming problem.

A topic with very little work done is the problem of optimum allocation in multivariate stratified random sampling when it is assumed that the variances in each stratum are unknown (the most common case in practice), in which case it would be needed to estimate these variances, see Cochran (1977), Thompson (1997) and Sukhatme *et al.* (1984). Under this context, the set of estimated variances for each stratum are random variables. This last situation has an important consequence when the optimum allocation problem in stratified random sampling is stated as a nonlinear mathematical programming. The univariate case for this setting was studied in detail by Díaz-García and Garay (2007). The present work owes some ideas to Melaku (1986) who studies the asymptotic normality of the optimal solution in multivariate stratified random sampling by taking into account the randomness of the set of estimated variances for each stratum.

The theory of *Stochastic Mathematical Programming* or *Stochastic Optimisation* is a

well established field of study that deals with the following general problem:

$$\begin{aligned} \min_{\mathbf{x}} \quad & \begin{pmatrix} h_1(\mathbf{x}, \boldsymbol{\xi}) \\ h_2(\mathbf{x}, \boldsymbol{\xi}) \\ \vdots \\ h_r(\mathbf{x}, \boldsymbol{\xi}) \end{pmatrix} \\ \text{subject to} \quad & g_j(\mathbf{x}, \boldsymbol{\xi}) \geq 0, \quad j = 1, 2, \dots, s, \end{aligned} \tag{1}$$

where \mathbf{x} is k -dimensional and $\boldsymbol{\xi}$ is m -dimensional. If \mathbf{x} or $\boldsymbol{\xi}$ are random, then (1) defines a multiobjective stochastic mathematical programming problem, see Prékopa (1995, p. 234) and Uryasev and Pardalos (2001). As shall be seen in the following sections, the optimum allocation problem in stratified random sampling can be proposed as a stochastic multiobjective mathematical programming problem.

This paper formulates the optimum allocation in multivariate stratified random sampling as a stochastic multiobjective integer mathematical programming problem constrained by a cost function or by a given sample size. Section 2 includes some notation and definitions on multivariate stratified random sampling under two approaches. Section 3 studies the asymptotic normality of the vector of sample variances. Two alternative approaches for solving the integer stochastic multiobjective mathematical programming problems are proposed in Section 4. In Section 5, under the first approach, diverse particular stochastic solutions are proposed. Similarly, several particular solutions are derived under the second suggested approach, see Section 6. Section 7 establish some equivalent deterministic programs to the stochastic solution stated in Section 6. Finally the techniques proposed are applied to an example of the literature, see Section 8.

2 Preliminary results on multivariate stratified random sampling

Assume a population of size N , divided into H sub-populations (strata). We wish to find a representative sample of size n and an optimum allocation in the strata, meeting the following requirements: i) to minimise the variance of the estimated mean subject to a budgetary constraint; or ii) to minimise the cost subject to a constraint on the variances; this is the classical problem in optimum allocation in univariate stratified sampling, see Cochran (1977), Sukhatme *et al.* (1984) and Thompson (1997). However, if more than one characteristic (variable) is being considered then the problem is known as optimum allocation in multivariate stratified sampling. For a formal expression of the problem of optimum allocation in stratified sampling, consider the following notation.

The subindex $h = 1, 2, \dots, H$ denotes the stratum, $i = 1, 2, \dots, N_h$ or n_h the unit within stratum h and $j = 1, 2, \dots, G$ denotes the characteristic (variable). Moreover:

N_h	Total number of units within stratum h
n_h	Number of units from the sample in stratum h
y_{hi}^j	Value obtained for the i -th unit in stratum h of the j -th characteristic

$\mathbf{n} = (n_1, \dots, n_H)'$	Vector of the number of units in the sample
$W_h = \frac{N_h}{N}$	Relative size of stratum h
$\bar{Y}_h^j = \frac{\sum_{i=1}^{N_h} y_{hi}^j}{N_h}$	Population mean in stratum h of the j -th characteristic
$\bar{\mathbf{Y}}_h = (\bar{Y}_h^1, \dots, \bar{Y}_h^G)'$	Population mean vector in stratum h
$\bar{y}_h^j = \frac{\sum_{i=1}^{n_h} y_{hi}^j}{n_h}$	Sample mean in stratum h of the j -th characteristic
$\bar{\mathbf{y}}_h = (\bar{y}_h^1, \dots, \bar{y}_h^G)'$	Sample mean vector in stratum h
$\bar{y}_{ST}^j = \sum_{h=1}^H W_h \bar{y}_h^j$	Estimator of the population mean in multivariate stratified sampling for the j -th characteristic
$\bar{\mathbf{y}}_{ST} = (\bar{y}_{ST}^1, \dots, \bar{y}_{ST}^G)'$	Estimator of the population mean vector in multivariate stratified sampling
S_{hj}^2	Population variance in stratum h
s_{hj}^2	Sample variance in stratum h
$\mathbb{S}_h = (S_{h1}^2, \dots, S_{hG}^2)'$	Vector of population variances in stratum h
$\mathbf{s}_h = (s_{h1}^2, \dots, s_{hG}^2)'$	Vector of variance estimators in stratum h
$\widehat{\text{Var}}(\bar{y}_{ST}^j)$	Estimated variance of \bar{y}_{ST}^j
c_h	Cost per G -dimensional sampling unit in stratum h and let $\mathbf{c} = (c_1, \dots, c_G)'$.

Define $\widehat{\mathbf{V}}_u(\bar{\mathbf{y}}_{ST}) \in \mathbb{R}^G$, as $\widehat{\mathbf{V}}_u(\bar{\mathbf{y}}_{ST}) = \left(\widehat{\text{Var}}(\bar{y}_{ST}^1), \dots, \widehat{\text{Var}}(\bar{y}_{ST}^G) \right)'$, where if $\mathbf{a} \in \mathbb{R}^G$, \mathbf{a}' denotes the transpose of \mathbf{a} . Finally, observe that $\widehat{\mathbf{V}}_u(\bar{\mathbf{y}}_{ST})$ is a function of \mathbf{n} , then it shall be written either $\widehat{\mathbf{V}}_u(\bar{\mathbf{y}}_{ST})$ or $\widehat{\mathbf{V}}_u(\bar{\mathbf{y}}_{ST}(\mathbf{n}))$. Similarly, it shall be written $\widehat{\text{Var}}(\bar{y}_{ST}^j)$ or $\widehat{\text{Var}}(\bar{y}_{ST}^j(\mathbf{n}))$.

3 Limiting distribution of the sample variance

In this section the asymptotic distribution of the estimator of the vector of variances \mathbf{s}_h is studied. With this aim in mind, the multivariate version of Hájek's theorem in the context of sampling theory is presented, see Hájek (1961). In what follows, from Lemma 3.1 through Theorem 3.1, asymptotic results are stated for a single stratum. The notation N_ν and n_ν denote the size of a generic stratum and the size of a simple random sample from that stratum.

Lemma 3.1. Let $\boldsymbol{\varrho}_\nu$ be a $G \times 1$ random vector defined as

$$\boldsymbol{\varrho}_\nu = \begin{bmatrix} \varrho_\nu^1 \\ \vdots \\ \varrho_\nu^G \end{bmatrix} = \frac{1}{n_\nu - 1} \left(\sum_{i=1}^{n_\nu} \left(y_{\nu i}^j - \bar{Y}_\nu^j \right)^2 \right)_{j=1, \dots, G}. \quad (2)$$

Assume that for $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_G)'$, any vector of constants,

$$\boldsymbol{\lambda}' (\mathbf{M}_\nu^4 - \mathbb{S}_\nu \mathbb{S}_\nu') \boldsymbol{\lambda} \geq \epsilon \max_{1 \leq j \leq G} \left[\lambda_j^2 \mathbf{e}_G^{j'} (\mathbf{M}_\nu^4 - \mathbb{S}_\nu \mathbb{S}_\nu') \mathbf{e}_G^j \right], \quad (3)$$

where $\mathbf{e}_G^j = (0, \dots, 0, 1, 0, \dots, 0)'$ is the j -th vector of the canonical base of \mathbb{R}^G , $\epsilon > 0$ and independent of $\nu > 1$ and

$$\mathbf{M}_\nu^4 = \frac{1}{N_\nu} \left(\sum_{i=1}^{N_\nu} \left(y_{\nu i}^k - \bar{Y}_\nu^k \right)^2 \left(y_{\nu i}^l - \bar{Y}_\nu^l \right)^2 \right)_{k, l=1, \dots, G}.$$

Suppose that $n_\nu \rightarrow \infty$, $N_\nu - n_\nu \rightarrow \infty$, $N_\nu \rightarrow \infty$, and that, for all $j = 1, \dots, G$,

$$\left[\lim_{\nu \rightarrow \infty} \left(\frac{n_\nu}{N_\nu} \right) = 0 \right] \Rightarrow \lim_{\nu \rightarrow \infty} \frac{\max_{1 \leq i_1 < \dots < i_{n_\nu} \leq N_\nu} \sum_{\beta=1}^{n_\nu} \left[\left(y_{\nu i_\beta}^j - \bar{Y}_\nu^j \right)^2 - S_{\nu j}^2 \right]^2}{N_\nu \left[m_{\nu j}^4 - \left(S_{\nu j}^2 \right)^2 \right]} = 0, \quad (4)$$

where

$$m_{\nu j}^4 = \frac{1}{N_\nu} \sum_{i=1}^{N_\nu} \left(y_{\nu i}^j - \bar{y}_\nu^j \right)^4.$$

Then, $\boldsymbol{\varrho}_\nu$ is asymptotically normally distributed as

$$\boldsymbol{\varrho}_\nu \xrightarrow{d} \mathcal{N}_G(\mathbf{E}(\boldsymbol{\varrho}_\nu), \text{Cov}(\boldsymbol{\varrho}_\nu)),$$

with

$$\mathbf{E}(\boldsymbol{\varrho}_\nu) = \frac{n_\nu}{n_\nu - 1} \mathbb{S}_\nu, \quad (5)$$

and

$$\text{Cov}(\boldsymbol{\varrho}_\nu) = \frac{n_\nu}{(n_\nu - 1)^2} (\mathbf{M}_\nu^4 - \mathbb{S}_\nu \mathbb{S}_\nu'). \quad (6)$$

n_ν is the sample size for a simple random sample from the ν -th population of size N_ν .

Remark 3.1. Let $\boldsymbol{\varrho}_\nu$ as in (2), taking $m = G$ and $a_{\nu i} = \left(y_{\nu i}^j - \bar{Y}_\nu^j \right)^2$ in Hájek (1961), it is obtained that:

i) $\boldsymbol{\varrho}_\nu$ can be expressed as

$$\boldsymbol{\varrho}_\nu = \left(\sum_{i=1}^{N_\nu} b_{\nu i} a_{\nu R_{\nu i}} \right)_{j=1, \dots, G}.$$

with b' s fixed, furthermore $b_{\nu 1} = \dots = b_{\nu n_\nu} = 1/(n_\nu - 1)$, $b_{\nu n_\nu + 1} = \dots = b_{\nu N_\nu} = 0$. Then

$$\lim_{\nu \rightarrow \infty} \frac{\max_{1 \leq j \leq N_\nu} (b_{\nu j} - \bar{b}_\nu)^2}{\sum_{i=1}^{N_\nu} (b_{\nu i} - \bar{b}_\nu)^2} = 0, \quad \text{where} \quad \bar{b}_\nu = \frac{1}{N_\nu} \sum_{i=1}^{N_\nu} b_{\nu i}$$

holds if $n_\nu \rightarrow \infty$, $N_\nu - n_\nu \rightarrow \infty$.

ii) \bar{a}_ν is

$$\begin{aligned} \bar{a}_\nu &= \frac{1}{N_\nu} \sum_{i=1}^{N_\nu} a_{\nu i} \\ &= \frac{1}{N_\nu} \sum_{i=1}^{N_\nu} (y_{\nu i}^j - \bar{Y}_\nu^j)^2 \\ &= S_{\nu j}^2 \end{aligned}$$

iii) From (7.2) in Hájek (1961)

$$\sum_{i=1}^{N_\nu} \left[\sum_{j=1}^k \lambda_j (a_{\nu i}^j - a_\nu^j) \right]^2 \geq \epsilon \max_{1 \leq j \leq k} \left[\lambda_j^2 \sum_{i=1}^{N_\nu} (a_{\nu i}^j - a_\nu^j)^2 \right]. \quad (7)$$

In the context of sampling theory, the right side in (7) can be written as

$$\begin{aligned} \sum_{i=1}^{N_\nu} \left[\sum_{j=1}^k \lambda_j (a_{\nu i}^j - a_\nu^j) \right]^2 &= \sum_{i=1}^{N_\nu} \left\{ \sum_{j=1}^k \lambda_j \left[(y_{\nu i}^j - \bar{Y}_\nu^j)^2 - S_{\nu j}^2 \right] \right\}^2 \\ &= \sum_{i=1}^{N_\nu} \left\{ \lambda' \begin{bmatrix} (y_{\nu i}^1 - \bar{Y}_\nu^1)^2 - S_{\nu 1}^2 \\ \vdots \\ (y_{\nu i}^G - \bar{Y}_\nu^G)^2 - S_{\nu G}^2 \end{bmatrix} \right\}^2 \\ &= \sum_{i=1}^{N_\nu} \lambda' \begin{bmatrix} (y_{\nu i}^1 - \bar{Y}_\nu^1)^2 - S_{\nu 1}^2 \\ \vdots \\ (y_{\nu i}^G - \bar{Y}_\nu^G)^2 - S_{\nu G}^2 \end{bmatrix} \\ &\quad \left[\left((y_{\nu i}^1 - \bar{Y}_\nu^1)^2 - S_{\nu 1}^2 \right), \dots, \left((y_{\nu i}^G - \bar{Y}_\nu^G)^2 - S_{\nu G}^2 \right) \right] \lambda \\ &= N_\nu \lambda' [m_{\nu kl}^4 - S_{\nu k}^2 S_{\nu l}^2]_{k,l=1,\dots,G} \lambda \\ &= N_\nu \lambda' (\mathbf{M}_\nu^4 - \mathbb{S}_\nu \mathbb{S}_\nu') \lambda, \end{aligned} \quad (8)$$

where \mathbf{M}_ν^4 is

$$= \frac{1}{N_\nu} \left[\sum_{i=1}^{N_\nu} (y_{\nu i}^k - \bar{Y}_\nu^k)^2 (y_{\nu i}^l - \bar{Y}_\nu^l)^2 \right]_{k,l=1,\dots,G}, \quad (9)$$

Similarly the right side of (7) is

$$\begin{aligned}\lambda_j^2 \sum_{i=1}^{N_\nu} (a_{\nu i}^j - a_\nu^j)^2 &= \sum_{i=1}^{N_\nu} \left\{ \lambda' \mathbf{e}_G^j \mathbf{e}_G^{j'} \left[(y_{\nu i}^j - \bar{Y}_\nu^j)^2 - S_{\nu j}^2 \right] \right\}^2 \\ &= \lambda_j^2 \sum_{i=1}^{N_\nu} \left\{ \mathbf{e}_G^{j'} \left[(y_{\nu i}^j - \bar{Y}_\nu^j)^2 - S_{\nu j}^2 \right] \right\}^2.\end{aligned}$$

Then, proceeding as in 3.,

$$\lambda_j^2 \sum_{i=1}^{N_\nu} (a_{\nu i}^j - a_\nu^j)^2 = N_\nu \lambda_j^2 \mathbf{e}_k^{j'} (\mathbf{M}_\nu^4 - \mathbb{S}_\nu \mathbb{S}_\nu') \mathbf{e}_k^j. \quad (10)$$

Therefore, from (8) and (10), (3) is established.

iv) The expression for (4) is found analogous to the procedure described in item 3.

v) Finally,

$$\begin{aligned}\mathbf{E}(\boldsymbol{\varrho}) &= \frac{1}{n_\nu - 1} \mathbf{E} \left(\sum_{i=1}^{n_\nu} (y_{\nu i}^j - \bar{Y}_\nu^j)^2 \right)_{j=1, \dots, G} \\ &= \frac{1}{n_\nu - 1} \left(\sum_{i=1}^{n_\nu} \mathbf{E} (y_{\nu i}^j - \bar{Y}_\nu^j)^2 \right)_{j=1, \dots, G} \\ &= \frac{1}{n_\nu - 1} \left(\sum_{i=1}^{n_\nu} S_{\nu j}^2 \right)_{j=1, \dots, G} \\ &= \frac{n_\nu}{n_\nu - 1} \mathbb{S}_\nu\end{aligned}$$

Similarly, by independence

$$\begin{aligned}\text{Cov}(\boldsymbol{\varrho}) &= \frac{1}{(n_\nu - 1)^2} \text{Cov} \left(\sum_{i=1}^{n_\nu} (y_{\nu i}^j - \bar{Y}_\nu^j)^2 \right)_{j=1, \dots, G} \\ &= \frac{1}{(n_\nu - 1)^2} \left\{ \mathbf{E} \left(\sum_{i=1}^{n_\nu} (y_{\nu i}^j - \bar{Y}_\nu^j)^2 \right)_{j=1, \dots, G} \left(\sum_{i=1}^{n_\nu} (y_{\nu i}^j - \bar{Y}_\nu^j)^2 \right)'_{j=1, \dots, G} \right. \\ &\quad \left. - \mathbf{E} \left(\sum_{i=1}^{n_\nu} (y_{\nu i}^j - \bar{Y}_\nu^j)^2 \right)_{j=1, \dots, G} \mathbf{E} \left(\sum_{i=1}^{n_\nu} (y_{\nu i}^j - \bar{Y}_\nu^j)^2 \right)'_{j=1, \dots, G} \right\} \\ &= \frac{n_\nu}{(n_\nu - 1)^2} (\mathbf{M}_\nu^4 - \mathbb{S}_\nu \mathbb{S}_\nu'),\end{aligned}$$

where \mathbf{M}_ν^4 is defined in (9).

Theorem 3.1. *Under assumptions in Lemma 3.1, the sequence of vector of sample variances \mathbf{s}_ν are such that \mathbf{s}_ν has an asymptotical normal with asymptotic mean and covariance matrix given by (5) and (6), respectively.*

Proof. This follows immediately from Lemma 3.1, only observe that

$$\begin{aligned}\mathfrak{s}_\nu &= \frac{1}{n_\nu - 1} \left(\sum_{i=1}^{n_\nu} \left(y_{\nu i}^j - \bar{y}_\nu^j \right)^2 \right)_{j=1, \dots, G} \\ &= \boldsymbol{\varrho} - \frac{n_\nu}{n_\nu - 1} \left((\bar{y}_\nu^j - \bar{Y}_\nu^j)^2 \right)_{j=1, \dots, G}\end{aligned}$$

where

$$\frac{n_\nu}{n_\nu - 1} \rightarrow 1 \quad \text{and} \quad (\bar{y}_\nu^j - \bar{Y}_\nu^j)^2 \rightarrow 0 \quad \text{in probability, } j = 1, \dots, G. \quad \square$$

□

As a direct consequence of Theorem 3.1 it is obtained:

Theorem 3.2. *Let $\widehat{\mathbf{V}}_u(\bar{\mathbf{y}}_{ST})$ be the estimator of the vector of variances of $\bar{\mathbf{y}}_{ST}$, then*

$$\widehat{\mathbf{V}}_u(\bar{\mathbf{y}}_{ST}) = \sum_{h=1}^H \left(\frac{W_h^2}{n_h} - \frac{W_h}{N} \right) \mathfrak{s}_h$$

is asymptotically normal distributed; moreover

$$\widehat{\mathbf{V}}_u(\bar{\mathbf{y}}_{ST}) \xrightarrow{d} \mathcal{N}_k \left(\mathbb{E} \left(\widehat{\mathbf{V}}_u(\bar{\mathbf{y}}_{ST}) \right), \text{Cov} \left(\widehat{\mathbf{V}}_u(\bar{\mathbf{y}}_{ST}) \right) \right),$$

where

$$\mathbb{E} \left(\widehat{\mathbf{V}}_u(\bar{\mathbf{y}}_{ST}) \right) = \sum_{h=1}^H \left(\frac{W_h^2}{n_h} - \frac{W_h}{N} \right) \frac{n_h}{n_h - 1} \mathbb{S}_h, \quad (11)$$

$$\text{Cov} \left(\widehat{\mathbf{V}}_u(\bar{\mathbf{y}}_{ST}) \right) = \sum_{h=1}^H \left(\frac{W_h^2}{n_h} - \frac{W_h}{N} \right) \frac{n_h}{(n_h - 1)^2} (\mathbf{M}_h^4 - \mathbb{S}_h \mathbb{S}_h'), \quad (12)$$

and

$$\mathbf{M}_h^4 = \frac{1}{N_h} \left[\sum_{i=1}^{N_h} \left(y_{hi}^k - \bar{Y}_h^k \right)^2 \left(y_{hi}^l - \bar{Y}_h^l \right)^2 \right]_{k, l=1, \dots, G},$$

n_ν is the sample size for a simple random sample from the ν -th population of size N_ν .

Observe that the asymptotic means and covariance matrices of the asymptotic normal distributions of \mathfrak{s}_h and $\widehat{\mathbf{V}}_u(\bar{\mathbf{y}}_{ST})$ are in terms of the population parameters $\bar{\mathbf{Y}}_h$, \mathbb{S}_h and \mathbf{M}_h^4 ; then, from Rao (1973), approximations of the asymptotic distribution are obtained making the following substitutions

$$\mathbb{S}_h \rightarrow \mathfrak{s}_h, \quad \text{and} \quad \mathbf{M}_h^4 \rightarrow \mathbf{m}_h^4 \quad (13)$$

where

$$\mathbf{m}_h^4 = (\widehat{m}_{hkl}^4)_{k, l=1, \dots, G} = \frac{1}{n_h} \left(\sum_{i=1}^{n_h} \left(y_{hi}^k - \bar{y}_h^k \right)^2 \left(y_{hi}^l - \bar{y}_h^l \right)^2 \right)_{k, l=1, \dots, G}.$$

4 Optimum allocation in multivariate stratified random sampling via multiobjective stochastic mathematical programming

When the variances are the objective functions, subject to certain cost function, the optimum allocation in multivariate stratified random sampling can be expressed as the following deterministic multiobjective mathematical programming

$$\begin{aligned} \min_{\mathbf{n}} \widehat{\mathbf{V}}_u(\bar{\mathbf{y}}_{ST}) &= \min_{\mathbf{n}} \begin{pmatrix} \widehat{\text{Var}}(\bar{y}_{ST}^1) \\ \vdots \\ \widehat{\text{Var}}(\bar{y}_{ST}^G) \end{pmatrix} \\ &\text{subject to} \\ \mathbf{c}'\mathbf{n} + c_0 &= C \\ 2 \leq n_h &\leq N_h, \quad h = 1, 2, \dots, H \\ n_h &\in \mathbb{N}, \end{aligned} \tag{14}$$

which has been studied in detail by Díaz-García and Ulloa (2008).

Observing that $\widehat{\mathbf{V}}_u(\bar{\mathbf{y}}_{ST})$ is a random variable then, as in (1), the optimum allocation via stochastic mathematical programming can be stated as the following stochastic multiobjective mathematical programming problem

$$\begin{aligned} \min_{\mathbf{n}} \widehat{\mathbf{V}}_u(\bar{\mathbf{y}}_{ST}) &= \min_{\mathbf{n}} \begin{pmatrix} \widehat{\text{Var}}(\bar{y}_{ST}^1) \\ \vdots \\ \widehat{\text{Var}}(\bar{y}_{ST}^G) \end{pmatrix} \\ &\text{subject to} \\ \mathbf{c}'\mathbf{n} + c_0 &= C \\ 2 \leq n_h &\leq N_h, \quad h = 1, 2, \dots, H \\ \widehat{\mathbf{V}}_u(\bar{\mathbf{y}}_{ST}) &\xrightarrow{d} \mathcal{N}_G \left(\mathbb{E} \left(\widehat{\mathbf{V}}_u(\bar{\mathbf{y}}_{ST}) \right), \text{Cov} \left(\widehat{\mathbf{V}}_u(\bar{\mathbf{y}}_{ST}) \right) \right) \\ n_h &\in \mathbb{N}, \end{aligned} \tag{15}$$

where $\mathbb{E} \left(\widehat{\mathbf{V}}_u(\bar{\mathbf{y}}_{ST}) \right)$ and $\text{Cov} \left(\widehat{\mathbf{V}}_u(\bar{\mathbf{y}}_{ST}) \right)$ are given by (11) and (12) respectively.

In the statistical context, two approaches can be proposed to solve problems of stochastic multiobjective mathematical programming of the type (15):

1. First, proposing a solution to the multiobjective mathematical programming problem, by converting it to an uniojective mathematical programming problem. Then this uniojective mathematical programming problem is solved as a stochastic programming problem by applying any of the techniques in the area. This approach has been applied in other areas of the Statistics, see Khuri and Cornell (1987) among others.
2. Given that methodologies exist, problems of the type (15) can be solved by applying directly the techniques of stochastic multiobjective mathematical programming, see Stancu-Minasian (1984) among others.

5 First approach

For this approach, the literature of multiobjective mathematical programming offers plenty of tools, see Díaz-García and Ulloa (2008), Ríos, Ríos Insua and Ríos Insua (1989), Miettinen (1999) and Steuer (1986). Here, the techniques of value function and goal programming are considered. Under the value function approach, (15) is formulated as

$$\begin{aligned}
& \min_{\mathbf{n}} \phi \left(\widehat{\mathbf{V}}_u(\overline{\mathbf{y}}_{ST}) \right), \\
& \text{subject to} \\
& \mathbf{c}'\mathbf{n} + c_0 = C \\
& 2 \leq n_h \leq N_h, \quad h = 1, 2, \dots, H \\
& \widehat{\mathbf{V}}_u(\overline{\mathbf{y}}_{ST}) \xrightarrow{d} \mathcal{N}_G \left(\mathbb{E} \left(\widehat{\mathbf{V}}_u(\overline{\mathbf{y}}_{ST}) \right), \text{Cov} \left(\widehat{\mathbf{V}}_u(\overline{\mathbf{y}}_{ST}) \right) \right) \\
& n_h \in \mathbb{N},
\end{aligned} \tag{16}$$

where $\phi(\cdot)$ is a value function¹ that summarises the importance of each of the variances of the G characteristics.

Similarly, in terms of goal programming (15) is stated as

$$\begin{aligned}
& \min_{\mathbf{n}} F(\mathbf{n}, \mathbb{S}) = \min_{\mathbf{x}} \sum_{j=1}^p w_j (d_j^+ + d_j^-) \\
& \text{subject to} \\
& \widehat{\text{Var}}(\overline{y}_{ST}^j(\mathbf{n})) + d_j^+ - d_j^- = t_j, \quad j = 1, \dots, G, \\
& \mathbf{c}'\mathbf{n} + c_0 = C \\
& 2 \leq n_h \leq N_h, \quad h = 1, 2, \dots, H \\
& \widehat{\mathbf{V}}_u(\overline{\mathbf{y}}_{ST}) \xrightarrow{d} \mathcal{N}_G \left(\mathbb{E} \left(\widehat{\mathbf{V}}_u(\overline{\mathbf{y}}_{ST}) \right), \text{Cov} \left(\widehat{\mathbf{V}}_u(\overline{\mathbf{y}}_{ST}) \right) \right) \\
& n_h \in \mathbb{N},
\end{aligned} \tag{17}$$

with

$$\begin{aligned}
d_j^+ &= \frac{1}{2} \left(\left| \widehat{\text{Var}}(\overline{y}_{ST}^j(\mathbf{n})) - t_j \right| + \left(\widehat{\text{Var}}(\overline{y}_{ST}^j(\mathbf{n})) - t_j \right) \right), \\
d_j^- &= \frac{1}{2} \left(\left| \widehat{\text{Var}}(\overline{y}_{ST}^j(\mathbf{n})) - t_j \right| - \left(\widehat{\text{Var}}(\overline{y}_{ST}^j(\mathbf{n})) - t_j \right) \right),
\end{aligned}$$

where the t_j 's are the target values for the objectives $\widehat{\text{Var}}(\overline{y}_{ST}^j)$, $j = 1, \dots, G$.

Note that, now, (16) and (17) are stochastic uniobjective mathematical programming problems, then, any technique from the area of stochastic programming could be applied. For example:

5.1 Modified expected value solution, E -model and V -model

Point $\mathbf{n} \in \mathbb{N}^H$ is the expected modified value solution to (16) if it is an efficient solution in the **Pareto**² sense to the following deterministic uniobjctive mathematical programming

¹A value function is a function $\phi : \mathbb{R}^H \rightarrow \mathbb{R}$ such that $\min \widehat{\mathbf{V}}_u(\overline{\mathbf{y}}_{ST}(\mathbf{n}^*)) < \min \widehat{\mathbf{V}}_u(\overline{\mathbf{y}}_{ST}(\mathbf{n}_1)) \Leftrightarrow \phi(\widehat{\mathbf{V}}_u(\overline{\mathbf{y}}_{ST}(\mathbf{n}^*))) < \phi(\widehat{\mathbf{V}}_u(\overline{\mathbf{y}}_{ST}(\mathbf{n}_1)))$, $\mathbf{n}^* \neq \mathbf{n}_1$.

²For matrix mathematical programming problems, there rarely exists a point $\mathbf{f}^*(\mathbf{n})$ which can be considered as the minimum. Alternatively, it said that $\mathbf{f}^*(\mathbf{n})$ is a *Pareto point* of $\mathbf{f}(\mathbf{n}) = (f_1(\mathbf{n}), \dots, f_G(\mathbf{n}))'$, if there is no other point $\mathbf{f}^1(\mathbf{n}^*)$ such that $\mathbf{f}^1(\mathbf{n}) \leq \mathbf{f}^*(\mathbf{n})$, i.e. for all j , $f_j(\mathbf{n}^1) \leq f_j(\mathbf{n}^*)$ and $\mathbf{f}^1(\mathbf{n}) \neq \mathbf{f}^*(\mathbf{n})$.

problem

$$\begin{aligned}
& \min_{\mathbf{n}} k_1 E \left(\phi \left(\widehat{\mathbf{V}}_u(\overline{\mathbf{y}}_{ST}) \right) \right) + k_2 \sqrt{\text{Var} \left(\phi \left(\widehat{\mathbf{V}}_u(\overline{\mathbf{y}}_{ST}) \right) \right)} \\
& \quad \text{subject to} \\
& \quad \mathbf{c}'\mathbf{n} + c_0 = C \\
& \quad 2 \leq n_h \leq N_h, \quad h = 1, 2, \dots, H \\
& \quad n_h \in \mathbb{N},
\end{aligned} \tag{18}$$

and for (17)

$$\begin{aligned}
& \min_{\mathbf{n}} k_1 E (F(\mathbf{n}, \mathbb{S})) + k_2 \sqrt{\text{Var} (F(\mathbf{n}, \mathbb{S}))} \\
& \quad \text{subject to} \\
& \quad \widehat{\text{Var}}(\overline{y}_{ST}^j(\mathbf{n})) + d_j^+ - d_j^- = t_j, \quad j = 1, \dots, G, \\
& \quad \mathbf{c}'\mathbf{n} + c_0 = C \\
& \quad 2 \leq n_h \leq N_h, \quad h = 1, 2, \dots, H \\
& \quad n_h \in \mathbb{N},
\end{aligned} \tag{19}$$

Here k_1 and k_2 are non negative constants, and their values show the relative importance of the expectation and the variance of $\phi \left(\widehat{\mathbf{V}}_u(\overline{\mathbf{y}}_{ST}) \right)$ and $F(\mathbf{n}, \mathbb{S})$. Some authors suggest that $k_1 + k_2 = 1$, see Rao (1979, p. 599). Observe that if we take $k_1 = 1$ and $k_2 = 0$ in (18), the resulting method is known as the E-model. Alternatively, if $k_1 = 0$ and $k_2 = 1$, the method is termed the V-model, see Charnes and Cooper (1963), Prékopa (1995) and Uryasev and Pardalos (2001).

5.2 Minimum risk solution of aspiration level τ , P-model

Point $\mathbf{n} \in \mathbb{N}^H$ is a minimum risk solution at the aspiration level τ to the problem (16) (also termed P-model by Charnes and Cooper (1963)) if it is an efficient solution in the Pareto sense of the uniobjective stochastic mathematical programming problem

$$\begin{aligned}
& \min_{\mathbf{n}} P \left(\phi \left(\widehat{\mathbf{V}}_u(\overline{\mathbf{y}}_{ST}) \right) \leq \tau \right) \\
& \quad \text{subject to} \\
& \quad \mathbf{c}'\mathbf{n} + c_0 = C \\
& \quad 2 \leq n_h \leq N_h, \quad h = 1, 2, \dots, H \\
& \quad n_h \in \mathbb{N}.
\end{aligned} \tag{20}$$

and correspondingly, for (17):

$$\begin{aligned}
& \min_{\mathbf{n}} P (F(\mathbf{n}, \mathbb{S}) \leq \tau_1) \\
& \quad \text{subject to} \\
& \quad \widehat{\text{Var}}(\overline{y}_{ST}^j(\mathbf{n})) + d_j^+ - d_j^- = t_j, \quad j = 1, \dots, G, \\
& \quad \mathbf{c}'\mathbf{n} + c_0 = C \\
& \quad 2 \leq n_h \leq N_h, \quad h = 1, 2, \dots, H \\
& \quad n_h \in \mathbb{N},
\end{aligned} \tag{21}$$

for an specified aspiration level τ_1 .

Among the multiobjective techniques we find that the value function method is, in general, the most commonly applied, its properties have been studied with most detail, see Ríos, Ríos Insua and Ríos Insua (1989), Miettinen (1999), Steuer (1986), and the references

therein. Note that the value function $\phi(\cdot)$ can be defined in an infinite number of forms, which represents a great obstacle for its definition. Fortunately, some simple functions have given excellent results in applications and they can be considered as promising approaches. One of these particular forms is the weighting method. Under this approach, problem (16) can be expressed as:

$$\begin{aligned}
& \min_{\mathbf{n}} \sum_{j=1}^G w_j \widehat{\text{Var}}(\bar{y}_{ST}^j), \\
& \text{subject to} \\
& \mathbf{c}'\mathbf{n} + c_0 = C \\
& 2 \leq n_h \leq N_h, \quad h = 1, 2, \dots, H \\
& \widehat{\mathbf{V}}_u(\bar{\mathbf{y}}_{ST}) \xrightarrow{d} \mathcal{N}_k \left(\mathbb{E} \left(\widehat{\mathbf{V}}_u(\bar{\mathbf{y}}_{ST}) \right), \text{Cov} \left(\widehat{\mathbf{V}}_u(\bar{\mathbf{y}}_{ST}) \right) \right) \\
& n_h \in \mathbb{N},
\end{aligned} \tag{22}$$

such that $\sum_{j=1}^G w_j = 1$, $w_j \geq 0 \quad \forall \quad j = 1, 2, \dots, G$; where the w_j 's weight the importance of each characteristic.

6 Second approach: Stochastic multiobjective mathematical programming approaches

In this section, solutions for problem (14) are proposed under diverse stochastic multiobjective mathematical programming approaches. The properties of the solution obtained under the different approaches are described in detail by Kataoka (1963), Stancu-Minasian (1984) and Prékopa (1995).

As shall be seen, each stochastic multiobjective mathematical programming approach can be stated in several ways. In some cases, these possibilities are a consequence of assuming whether $\widehat{\text{Var}}(\bar{y}_{ST}^j)$ $j = 1, \dots, G$, are correlated or not.

6.1 Multiobjective expected value solution, multiobjective E-model

Point $\mathbf{n} \in \mathbb{N}^G$ is the expected value solution to (14) if it is an efficient solution in the Pareto sense to the following deterministic multiobjective mathematical programming problem

$$\begin{aligned}
& \min_{\mathbf{n}} \mathbb{E} \left(\widehat{\mathbf{V}}_u(\bar{\mathbf{y}}_{ST}) \right), \\
& \text{subject to} \\
& \mathbf{c}'\mathbf{n} + c_0 = C \\
& 2 \leq n_h \leq N_h, \quad h = 1, 2, \dots, H \\
& n_h \in \mathbb{N}.
\end{aligned} \tag{23}$$

6.2 Multiobjective minimum variance solution, multiobjective V-model

The point $\mathbf{n} \in \mathbb{N}^G$ is the minimum variance solution to the problem (14) if it is an efficient solution in the Pareto sense of the deterministic multiobjective mathematical programming

problem

$$\begin{aligned}
& \min_{\mathbf{n}} \begin{pmatrix} \text{Var} \left(\widehat{\text{Var}}(\bar{y}_{ST}^1) \right) \\ \vdots \\ \text{Var} \left(\widehat{\text{Var}}(\bar{y}_{ST}^G) \right) \end{pmatrix} \\
& \text{subject to} \\
& \mathbf{c}'\mathbf{n} + c_0 = C \\
& 2 \leq n_h \leq N_h, \quad h = 1, 2, \dots, H \\
& n_h \in \mathbb{N},
\end{aligned} \tag{24}$$

This efficient solution is adequate if it is assumed that $\widehat{\text{Var}}(\bar{y}_{ST}^j)$, $j = 1, \dots, G$ are uncorrelated, however if they are correlated then a more adequate approach is:

The point $\mathbf{n} \in \mathbb{N}^G$ is the minimum variance solution to the problem (14) if it is an efficient solution in the Pareto sense of the deterministic matrix mathematical programming problem

$$\begin{aligned}
& \min_{\mathbf{n}} \text{Cov} \left(\widehat{\mathbf{V}}_u(\bar{\mathbf{y}}_{ST}) \right) \\
& \text{subject to} \\
& \mathbf{c}'\mathbf{n} + c_0 = C \\
& 2 \leq n_h \leq N_h, \quad h = 1, 2, \dots, H \\
& n_h \in \mathbb{N}.
\end{aligned} \tag{25}$$

A general approach is studied by Díaz García and Ramos-Quiroga (2001).

6.3 Multiobjective expected value standard deviation solution, multiobjective modified E-model

Point $\mathbf{n} \in \mathbb{N}^G$ is an expected value standard deviation solution to the problem (14) if it is an efficient solution in the Pareto sense of the mixed deterministic multiobjective-matrix mathematical programming problem

$$\begin{aligned}
& \min_{\mathbf{n}} \left[\begin{array}{c} \text{E} \left(\widehat{\mathbf{V}}_u(\bar{\mathbf{y}}_{ST}) \right) \\ \left(\text{Cov} \left(\widehat{\mathbf{V}}_u(\bar{\mathbf{y}}_{ST}) \right) \right)^{1/2} \end{array} \right] \\
& \text{subject to} \\
& \mathbf{c}'\mathbf{n} + c_0 = C \\
& 2 \leq n_h \leq N_h, \quad h = 1, 2, \dots, H \\
& n_h \in \mathbb{N}.
\end{aligned} \tag{26}$$

where $(\mathbf{A}^{1/2})^2 = \mathbf{A}$, see Muirhead (1982, Appendix).

We now define the concept of efficient solution multiobjective minimum risk of joint aspiration level $\boldsymbol{\tau} = (\tau_1, \tau_2, \dots, \tau_G)'$ and the efficient solution with joint probability α . Both solutions are obtained, respectively, by applying the multivariate versions of minimum risk and Kataoka criteria, referred to in the literature as criteria of maximum probability or satisfying criteria, due to the fact that, as we shall see, in both cases the criteria to be used provide, in one way or another, “good” solutions in terms of probability, see Kataoka (1963).

6.4 Multiobjective minimum risk solution of joint aspiration level τ , multiobjective modified P-model

Point $\mathbf{n} \in \mathbb{N}^G$ is a minimum risk solution at joint aspiration level τ to the problem (14) if it is an efficient solution in the Pareto sense of the multiobjective stochastic mathematical programming problem

$$\begin{aligned} \min_{\mathbf{n}} & \begin{pmatrix} P\left(\widehat{\text{Var}}(\bar{y}_{ST}^1) < \tau_1\right) \\ \vdots \\ P\left(\widehat{\text{Var}}(\bar{y}_{ST}^G) < \tau_G\right) \end{pmatrix} \\ & \text{subject to} \\ & \mathbf{c}'\mathbf{n} + c_0 = C \\ & 2 \leq n_h \leq N_h, \quad h = 1, 2, \dots, H \\ & n_h \in \mathbb{N}. \end{aligned} \tag{27}$$

It is also possible consider the follow alternative multiobjective P-model

$$\begin{aligned} \min_{\mathbf{n}} & P \begin{pmatrix} \widehat{\text{Var}}(\bar{y}_{ST}^1) < \tau_1 \\ \vdots \\ \widehat{\text{Var}}(\bar{y}_{ST}^G) < \tau_G \end{pmatrix} \\ & \text{subject to} \\ & \mathbf{c}'\mathbf{n} + c_0 = C \\ & 2 \leq n_h \leq N_h, \quad h = 1, 2, \dots, H \\ & n_h \in \mathbb{N}. \end{aligned} \tag{28}$$

Again, (28) is more adequate if the response variables are correlated. However (28) is sensibly more complicated to solve than (27). When $G = 2$, Prékopa (1970) proposed an algorithm in a similar problem (probabilistic constrained programming), which can be apply to solve (28).

6.5 Multiobjective Kataoka solution with probability α

Point $\mathbf{n} \in \mathbb{N}^G$ is a multiobjective Kataoka solution with probability α (fixed) to the problem (14) if it is an efficient solution in the Pareto sense of the multiobjective mathematical programming problem

$$\begin{aligned} \min_{\mathbf{n}, \tau} & \tau \\ & \text{subject to} \\ & P\left(\widehat{\text{Var}}(\bar{y}_{ST}^j) \leq \tau_j\right) = \alpha, \quad j = 1, \dots, G \\ & \mathbf{c}'\mathbf{n} + c_0 = C \\ & 2 \leq n_h \leq N_h, \quad h = 1, 2, \dots, H \\ & n_h \in \mathbb{N}. \end{aligned} \tag{29}$$

Alternatively (29) can be proposed as

$$\begin{aligned}
& \min_{\mathbf{n}, \boldsymbol{\tau}} \boldsymbol{\tau} \\
& \text{subject to} \\
& \mathbf{P} \left(\begin{array}{c} \widehat{\text{Var}}(\bar{y}_{ST}^1) \leq \tau_1 \\ \vdots \\ \widehat{\text{Var}}(\bar{y}_{ST}^G) \leq \tau_r \end{array} \right) = \alpha \\
& \mathbf{c}'\mathbf{n} + c_0 = C \\
& 2 \leq n_h \leq N_h, \quad h = 1, 2, \dots, H \\
& n_h \in \mathbb{N}.
\end{aligned} \tag{30}$$

Note that (29) and (30) are multiobjective probabilistic constrained programs, see Charnes and Cooper (1963), Stancu-Minasian (1984) and Prékopa (1995).

Many others approaches can be used to solve (14). For example, Stancu-Minasian (1984) propose a stochastic version of a sequential technique termed Lexicographic method, for solving (24) and (27), or to applying it directly to (14); among many others options.

7 Equivalent deterministic programs

In this section we study in detail several particular deterministic programs equivalent to the stochastic multiobjective program (15). Consider first the following remark: from (11), (12), (13) and $j = 1, \dots, G$,

$$\begin{aligned}
\widehat{\mathbf{E}} \left(\widehat{\text{Var}}(\bar{y}_{ST}^j) \right) &= \sum_{h=1}^H \left(\frac{W_h^2}{n_h} - \frac{W_h}{N} \right) \frac{n_h}{n_h - 1} s_{hj}^2, \\
\widehat{\text{Var}} \left(\widehat{\text{Var}}(\bar{y}_{ST}^j) \right) &= \sum_{h=1}^H \left(\frac{W_h^2}{n_h} - \frac{W_h}{N} \right) \frac{n_h}{(n_h - 1)^2} (m_{hj}^4 - (s_{hj}^2)^2),
\end{aligned}$$

and

$$\widehat{m}_{hj}^4 = \frac{1}{n_h} \sum_{i=1}^{n_h} (y_{hi}^j - \bar{Y}_h^j)^4.$$

7.1 Multiobjective V -model

From (24) the multiobjective deterministic problem equivalent to (15) via the V -model is

$$\begin{aligned}
& \min_{\mathbf{n}} \left(\begin{array}{c} \widehat{\text{Var}} \left(\widehat{\text{Var}}(\bar{y}_{ST}^1) \right) \\ \vdots \\ \widehat{\text{Var}} \left(\widehat{\text{Var}}(\bar{y}_{ST}^G) \right) \end{array} \right) \\
& \text{subject to} \\
& \mathbf{c}'\mathbf{n} + c_0 = C \\
& 2 \leq n_h \leq N_h, \quad h = 1, 2, \dots, H \\
& n_h \in \mathbb{N}.
\end{aligned} \tag{31}$$

Which is solved by applying any of the multiobjective mathematical programming techniques.

7.2 Multiobjective P-model

Proceeding as in Díaz García *et al.* (2005), the multiobjective deterministic problem equivalent to (15) via the P-model (27) is

$$\begin{aligned} \min_{\mathbf{n}} \quad & \begin{pmatrix} \frac{\tau_1 - \widehat{\mathbb{E}}\left(\widehat{\text{Var}}(\overline{y}_{ST}^1)\right)}{\sqrt{\widehat{\text{Var}}\left(\widehat{\text{Var}}(\overline{y}_{ST}^1)\right)}} \\ \vdots \\ \frac{\tau_G - \widehat{\mathbb{E}}\left(\widehat{\text{Var}}(\overline{y}_{ST}^G)\right)}{\sqrt{\widehat{\text{Var}}\left(\widehat{\text{Var}}(\overline{y}_{ST}^G)\right)}} \end{pmatrix} \\ \text{subject to} \quad & \mathbf{c}'\mathbf{n} + c_0 = C \\ & 2 \leq n_h \leq N_h, \quad h = 1, 2, \dots, H \\ & n_h \in \mathbb{N}. \end{aligned} \tag{32}$$

7.3 Multiobjective Kataoka model

From Díaz García *et al.* (2005), the multiobjective deterministic problem equivalent to (15) via the Kataoka model (29) is given by

$$\begin{aligned} \min_{\mathbf{n}} \quad & \begin{pmatrix} \widehat{\mathbb{E}}\left(\widehat{\text{Var}}(\overline{y}_{ST}^1)\right) + \Phi^{-1}(\delta) \sqrt{\widehat{\text{Var}}\left(\widehat{\text{Var}}(\overline{y}_{ST}^1)\right)} \\ \vdots \\ \widehat{\mathbb{E}}\left(\widehat{\text{Var}}(\overline{y}_{ST}^G)\right) + \Phi^{-1}(\delta) \sqrt{\widehat{\text{Var}}\left(\widehat{\text{Var}}(\overline{y}_{ST}^G)\right)} \end{pmatrix} \\ \text{subject to} \quad & \mathbf{c}'\mathbf{n} + c_0 = C \\ & 2 \leq n_h \leq N_h, \quad h = 1, 2, \dots, H \\ & n_h \in \mathbb{N}. \end{aligned} \tag{33}$$

where Φ denotes the distribution function of the standard Normal distribution.

Similar multiobjective deterministic problems equivalent to (15) are obtained applying the other stochastic solutions described in Section 4. Note that, if we combine each stochastic solution with each multiobjective optimisation technique, we obtain an infinite number of possible solutions of (15). For example, note that the function of value $f(\cdot)$ may take an infinite number of forms. One of these particular forms is the weighting method, see Ríos, Ríos Insua and Ríos Insua (1989) and Steuer (1986). Under this approach, problem (33) can be solved as:

$$\begin{aligned} \min_{\mathbf{n}} \quad & \sum_{j=1}^G w_j \left\{ \widehat{\mathbb{E}}\left(\widehat{\text{Var}}(\overline{y}_{ST}^j)\right) + \Phi^{-1}(\delta) \sqrt{\widehat{\text{Var}}\left(\widehat{\text{Var}}(\overline{y}_{ST}^j)\right)} \right\} \\ \text{subject to} \quad & \mathbf{c}'\mathbf{n} + c_0 = C \\ & 2 \leq n_h \leq N_h, \quad h = 1, 2, \dots, H \\ & n_h \in \mathbb{N}. \end{aligned} \tag{34}$$

such that $\sum_{j=1}^G w_j = 1$, $w_j \geq 0 \forall j = 1, 2, \dots, G$: where w_j weights the importance of each characteristic. This solution can be termed the multiobjective Kataoka-weighting solution with probability α equivalent to the problem (15), via the weighting method.

Similarly, the multiobjective Kataoka solution with probability α to the problem (14), via the goal programming is

$$\begin{aligned} & \min_{\mathbf{n}} \sum_{j=1}^G w_j (d_j^+ + d_j^-) \\ & \text{subject to} \\ & \widehat{\mathbf{E}} \left(\widehat{\text{Var}}(\bar{y}_{ST}^j) \right) + \Phi^{-1}(\delta) \sqrt{\widehat{\text{Var}} \left(\widehat{\text{Var}}(\bar{y}_{ST}^j) \right)} - d_j^+ + d_j^- = \tau_j, \quad j = 1, \dots, G, \\ & \mathbf{c}'\mathbf{n} + c_0 = C \\ & 2 \leq n_h \leq N_h, \quad h = 1, 2, \dots, H \\ & n_h \in \mathbb{N}. \end{aligned} \tag{35}$$

where

$$\begin{aligned} d_j^+ &= \frac{1}{2} \left(\left| \widehat{\mathbf{E}} \left(\widehat{\text{Var}}(\bar{y}_{ST}^j) \right) + \Phi^{-1}(\delta) \sqrt{\widehat{\text{Var}} \left(\widehat{\text{Var}}(\bar{y}_{ST}^j) \right)} - \tau_j \right| \right. \\ & \quad \left. + \left(\widehat{\mathbf{E}} \left(\widehat{\text{Var}}(\bar{y}_{ST}^j) \right) + \Phi^{-1}(\delta) \sqrt{\widehat{\text{Var}} \left(\widehat{\text{Var}}(\bar{y}_{ST}^j) \right)} - \tau_j \right) \right), \\ d_j^- &= \frac{1}{2} \left(\left| \widehat{\mathbf{E}} \left(\widehat{\text{Var}}(\bar{y}_{ST}^j) \right) + \Phi^{-1}(\delta) \sqrt{\widehat{\text{Var}} \left(\widehat{\text{Var}}(\bar{y}_{ST}^j) \right)} - \tau_j \right| \right. \\ & \quad \left. - \left(\widehat{\mathbf{E}} \left(\widehat{\text{Var}}(\bar{y}_{ST}^j) \right) + \Phi^{-1}(\delta) \sqrt{\widehat{\text{Var}} \left(\widehat{\text{Var}}(\bar{y}_{ST}^j) \right)} - \tau_j \right) \right). \end{aligned}$$

Note that so far, the cost constraint $\sum_{h=1}^H c_h n_h + c_0 = C$ has been used in every stochastic multiobjective mathematical programming method. However, in several situations, it can be used to represent existing restrictions for the availability of man-hours for carrying out a survey, or restrictions on the total available time for performing the survey, etc. These cases can be implemented by using the following constraint, see Arthanari and Dodge (1981):

$$\sum_{h=1}^H n_h = n.$$

8 Application

Consider data from a case study presented by Arvanitis and Afonja (1971). They study a forest survey conducted in Humbolt County, California. The population was subdivided into nine strata on the basis of the timber volume per unit area, as determined from aerial photographs. The two variables included in this example are the basal area³, (BA), in square feet, and the net volume in cubic feet (Vol.), both expressed on a per acre basis. The variances, covariances and the number of units within stratum h are listed in Table 1.

³In forestry terminology, 'Basal area' is the area of a plant perpendicular to the longitudinal axis of a tree at 4.5 feet above ground.

Table 1: Variances, covariances and the number of units within each stratum

Stratum	N_h	Variance		Covariance
		BA	Vol.	
1	11 131	1 557	554 830	28 980
2	65 857	3 575	1 430 600	61 591
3	106 936	3 163	1 997 100	72 369
4	72 872	6 095	5 587 900	166 120
5	78 260	10 470	10 603 000	293 960
6	51 401	8 406	15 828 000	357 300
7	24 050	20 115	26 643 000	663 300
8	46 113	9 718	13 603 000	346 810
9	102 985	2 478	1 061 800	39 872

In this example, the stochastic multiobjective mathematical programming problem under approach (15) is

$$\begin{aligned}
& \min_{\mathbf{n}} \begin{pmatrix} \widehat{\text{Var}}(\bar{y}_{ST}^1) \\ \widehat{\text{Var}}(\bar{y}_{ST}^2) \end{pmatrix} \\
& \text{subject to} \\
& \sum_{h=1}^9 n_h = 1000 \\
& \widehat{\mathbf{V}}_u(\bar{\mathbf{y}}_{ST}) \xrightarrow{d} \mathcal{N}_2 \left(\mathbb{E} \left(\widehat{\mathbf{V}}_u(\bar{\mathbf{y}}_{ST}) \right), \text{Cov} \left(\widehat{\mathbf{V}}_u(\bar{\mathbf{y}}_{ST}) \right) \right) \\
& 2 \leq n_h \leq N_h, \quad h = 1, 2, \dots, 9 \\
& n_h \in \mathbb{N},
\end{aligned} \tag{36}$$

where $\mathbb{E} \left(\widehat{\mathbf{V}}_u(\bar{\mathbf{y}}_{ST}) \right)$ and $\text{Cov} \left(\widehat{\mathbf{V}}_u(\bar{\mathbf{y}}_{ST}) \right)$ are given by (11) and (12) respectively.

8.1 Particular solutions under the first approach

Taking into account (11), (12) and (13), and from (18) and (22) the equivalent deterministic problem to (36) is

$$\begin{aligned}
& \min_{\mathbf{n}} \left\{ \begin{aligned} & k_1 \left(\sum_{j=1}^2 w_j \widehat{\mathbb{E}} \left[\widehat{\text{Var}}(\bar{y}_{ST}^j) \right] \right) \\ & + k_2 \left(\sqrt{\sum_{j=1}^2 w_j^2 \widehat{\text{Var}} \left[\widehat{\text{Var}}(\bar{y}_{ST}^j) \right] + 2w_1 w_2 \widehat{\text{Cov}} \left(\widehat{\text{Var}}(\bar{y}_{ST}^1), \widehat{\text{Var}}(\bar{y}_{ST}^2) \right)} \right) \end{aligned} \right\} \\
& \text{subject to} \\
& \sum_{h=1}^9 n_h = 1000 \\
& 2 \leq n_h \leq N_h, \quad h = 1, 2, \dots, 9 \\
& n_h \in \mathbb{N},
\end{aligned}$$

where $w_1 + w_2 = 1$, and $k_1 + k_2 = 1$ $w_j, r_j(\text{fixed}) \geq 0 \quad \forall \quad j = 1, 2$, whose solution is termed: the *weighting-modified E-model solution*.

Similarly, from (21), the *weighting-P-model solution* of (36) is obtained solving the following equivalent deterministic problem

$$\begin{aligned}
& \min_{\mathbf{n}} \frac{\tau - \sum_{j=1}^2 w_j \widehat{\mathbb{E}} \left[\widehat{\text{Var}}(\overline{y}_{ST}^j) \right]}{\sqrt{\sum_{j=1}^2 w_j^2 \widehat{\text{Var}} \left[\widehat{\text{Var}}(\overline{y}_{ST}^j) \right] + 2w_1 w_2 \widehat{\text{Cov}} \left(\widehat{\text{Var}}(\overline{y}_{ST}^1), \widehat{\text{Var}}(\overline{y}_{ST}^2) \right)}} \\
& \text{subject to} \\
& \sum_{h=1}^9 n_h = 1000 \\
& 2 \leq n_h \leq N_h, \quad h = 1, 2, \dots, 9 \\
& n_h \in \mathbb{N},
\end{aligned}$$

8.2 Particular solutions via the second approach

From Section 7, using the weighting method, the *multiobjective V-model-weighting model solution* of (36) is obtained from the following equivalent deterministic problem

$$\begin{aligned}
& \min_{\mathbf{n}} \sum_{j=1}^2 w_j \left\{ \widehat{\text{Var}} \left(\widehat{\text{Var}}(\overline{y}_{ST}^j) \right) \right\} \\
& \text{subject to} \\
& \sum_{h=1}^9 n_h = 1000 \\
& 2 \leq n_h \leq N_h, \quad h = 1, 2, \dots, 9 \\
& n_h \in \mathbb{N}.
\end{aligned}$$

Analogously, the *multiobjective P-model-weighting model solution* of (36) is obtained by solving the following equivalent deterministic problem

$$\begin{aligned}
& \min_{\mathbf{n}} \sum_{j=1}^2 w_j \frac{\tau_j - \widehat{\mathbb{E}} \left(\widehat{\text{Var}}(\overline{y}_{ST}^j) \right)}{\sqrt{\widehat{\text{Var}} \left(\widehat{\text{Var}}(\overline{y}_{ST}^j) \right)}} \\
& \text{subject to} \\
& \sum_{h=1}^9 n_h = 1000 \\
& 2 \leq n_h \leq N_h, \quad h = 1, 2, \dots, 9 \\
& n_h \in \mathbb{N}.
\end{aligned}$$

Similarly, solving the following equivalent deterministic problem, the *multiobjective Kataoka-*

weighting solution of (36) is obtained

$$\begin{aligned} \min_{\mathbf{n}} \sum_{j=1}^2 w_j \left\{ \widehat{\mathbb{E}} \left(\widehat{\text{Var}}(\bar{y}_{ST}^j) \right) + \Phi^{-1}(\delta) \sqrt{\widehat{\text{Var}} \left(\widehat{\text{Var}}(\bar{y}_{ST}^j) \right)} \right\} \\ \text{subject to} \\ \sum_{h=1}^9 n_h = 1000 \\ 2 \leq n_h \leq N_h, \quad h = 1, 2, \dots, 9 \\ n_h \in \mathbb{N}. \end{aligned}$$

In all cases, $\sum_{j=1}^G w_j = 1$, $w_j \geq 0 \forall j = 1, 2$: where the w_j 's weight the importance of each characteristic.

Table 2 shows the solutions obtained by some of the methods described in Sections 5, in particular, the weighting-modified E -model, weighting- E -model and weighting- V -model are presented. Also, the solutions described in Section 7, are included, specifically the multiobjective V -model-weighting, P -model-weighting and Kataoka model-weighting models. The first two rows in Table 2 include the optimum allocation for each characteristic, BA and Vol. The last two columns show the minimum values of the individual variances for the respective optimum allocations identified by each method. The results were computed using the commercial software Hyper LINGO/PC, release 6.0, see Winston (1995). The default mathematical programming methods used by LINGO to solve the nonlinear integer mathematical programming programs are Generalised Reduced Gradient (GRG) and branch-and-bound methods, see Bazaraa *et al.* (2006). Some technical details of the computations are the following: The maximum number of iterations of the methods presented in Table 2 was 2364 (multiobjective Kataoka-weighting solution) and the mean execution time for all the programs was 2 seconds. Finally, observe that the greatest discrepancy found by the different methods among the sizes of the strata occurred under the multiobjective P -model-weighting solution. In part, this discrepancy is a consequence of the particular values for τ_1 and τ_2 in this method.

Conclusions

It is important to stress that there is a potentially infinite number of possible solutions to a stochastic multiobjective mathematical programming problem. Simply note that such an infinite number due to the flexibility in choosing the value function. Therefore, this paper presents only some few techniques of the area of multiobjective, stochastic and stochastic multiobjective mathematical programming, which when combined, produce a number of possible solutions to the problem of optimum allocation in multivariate stratified random sampling from a stochastic multiobjective mathematical programming point of view.

Because of all these possibilities, it is difficult to set general rules for how to use these techniques, leaving all responsibility on the skill of an expert in the area to determine which of these approaches is best in a particular application.

As the reader can see, several solutions are given in the context of the application, and, for the sake of completeness, the solutions in terms of goal programming are only indicated, see Kazemzadeh *et al.* (2008).

Table 2: Sample sizes and estimator of variances for the different allocations calculated

Allocation	n_1	n_2	n_3	n_4	n_5	n_6	n_7	n_8	n_9	$\widehat{\text{Var}}(\bar{y}_{ST}^1)$	$\widehat{\text{Var}}(\bar{y}_{ST}^2)$
BA	10	94	144	136	191	113	81	109	122	5.591	5441.105
Vol	7	62	119	136	200	161	98	134	83	5.953	5139.531
First approach^a											
Weighting-modified E -model	8	46	77	119	191	191	158	161	49	7.311	5593.494
Weighting- E -model	7	63	119	135	200	160	98	134	84	5.936	5139.645
Weighting- V -model	8	46	77	119	191	121	158	161	49	7.526	5997.963
Second approach^b											
Multiobjective											
V -model-weighting model	8	46	77	119	191	191	158	161	49	7.311	5593.494
Multiobjective											
P -model-weighting model ^c	247	42	34	77	127	106	221	117	29	11.817	8980.960
Multiobjective											
Kataoka-weighting model ^d	8	46	77	119	191	191	158	161	49	7.311	5593.494

^aWith $k_1 = k_2 = 0.5$ and $w_1 = w_2 = 0.5$.

^bWhere $w_1 = w_2 = 0.5$.

^cWhere $\tau_1 = 6$ and $\tau_2 = 6000$.

^dWith $\Phi^{-1}(\delta) = 1.645$

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